Efficient Design of an Automated Market Maker*

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Abstract

Standard liquidity pools must charge relatively large fees to compensate liquidity providers for the impermanent loss. This induces price-staleness and the large fees tend to make effective AMM prices unattractive to traders, as better effective prices are typically available on centralized exchanges. In an idealized framework, with informed and uninformed traders, we show that auctioning an exclusive right to be a zero-fee trader can greatly improve the price efficiency in AMM. Moreover, this novel approach has the potential to substantially increase the profits of liquidity pools.

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1 Introduction

Since the introduction of Uniswap in 2020, Automated Market Makers (AMMs) have become a significant part of the cryptocurrency trading landscape. AMMs rose in popularity because of their ability to facilitate trades using considerably fewer blockchain resources (i.e. gas) and for their ability to provide deep liquidity, especially when compared to order books. However, current AMM systems face several challenges.

First, AMMs struggle making liquidity provision profitable. Changes in the balances of underlying tokens create Impermanent Loss for liquidity providers, and aggregate trading fees earned by the pool must be considerable to overcome this loss, as reported by Kim [2023]. Second, because AMMs are not directly connected to external markets they often struggle to maintain price synchronization with those markets[1]. The cost of trading creates a lag in the AMM price as arbitrageurs will only place trades in an AMM when their profits are sufficiently large enough to overcome the cost of transacting (e.g. gas, fees, and slippage). Since the cost of transacting on an AMM is often larger than on other markets this reduces trading volume, thereby reducing the amount of fees collected. This demonstrates a potential negative feedback loop in many current AMM designs. Profits for Liquidity Providers are a function of fees overcoming Impermanent Loss. If there is friction for arbitrageurs, Liquidity Providers experience reduced trading volumes and revenues, and may leave the pool thereby reducing the future amount of arbitrage.

Ripple's research team has put forth an innovative solution to address the inefficiencies seen in AMMs: a continuous auction mechanism designed to enhance the price responsiveness of AMMs. Specifically, users can bid LP tokens to hold a slot (Auction Slot Mechanism), which gives the holder zero-fee trades in the AMM, especially Constant Product Market Makers(CPMM). This paper delves into the Auction Slot Mechanism and assesses its cost and benefit to both arbitrageurs and LPs.

In section 2 of the paper we derive the expected profit for monopoly zero-fee trader. In section 3 we describe the continuous auction mechanism and derive the equilibrium price for the slot to be a monopoly zero-fee trader. Finally, in conclusion, we discuss potential shortcomings of the auction mechanism under more realistic assumptions. In Appendix, we propose an algorithm to find the the optimal bidding utilizing Monte Carlo method in Reinforcement Learning.

2 Expected Profit for Monopoly Zero-Fee Trader

We begin by deriving the expected profit for a zero-fee trader who does not face competition from other zero-fee traders. Here profit will differ

[1] Hansen, Kim, and Kimbrough [2022] documents that price formation does not take place on AMMs and therefore AMMs require arbitrageurs to keep the pool's price in line with the true price. Furthermore, the need for arbitrageurs to create price alignment is a design feature in an AMM system.

from the expected profit for a slot holder, because the latter must factor in the risk of being outbid by a competitor, and thereby losing the slot. We return to this aspect of auction competition in the last subsection.

A zero-fee trader can realize profit whenever there is a discrepancy between the marginal price offered by the liquidity pool[2] and the best available price on other platforms. We denote the marginal price by P and we use P^* to introduce a *true price*. The true price is characterized by having unpredictable price changes[3] and it plays the role of the best available price. Gaps between the true price and marginal price create statistical arbitrage for a zero-fee trader.

Price gaps may arise either from movements in the true price or shifts in the marginal price. We differentiate between these two sources of arbitrage profit, calling arbitrage initiated by the shift in true price as *volatility arbitrage* (Π^{Vol}) and arbitrage initiated by the shift in the marginal price as *noise trader arbitrage* (Π^{NT}). The sum of the two sources of arbitrage defines the arbitrage profit, $\Pi^{\text{Arb}} = \Pi^{\text{Vol}} + \Pi^{\text{NT}}$.

Let $\mathbb{E}_t$ denote the expectation conditional on all variables observed at time t. Then the instantaneous arbitrage profit,

$$\pi_t \equiv \lim_{h \to 0} \frac{\mathbb{E}_t\left[\Pi^{\text{Arb}}_{t,t+h}\right]}{h},$$

equals

$$\pi_t = \pi(Y_t, \sigma_t^2) = \sigma_t^2 \frac{Y_t}{4} + \lambda \frac{\omega^2}{Y_t}, \tag{1}$$

where Y_t is the value of numéraire in a liquidity pool at time t, σ_t is the volatility of the true price at time t, and λ is the arrival intensity of noise traders. The two terms on the right hand side of (1), $\sigma_t^2 Y_t/4$ and $\lambda\omega^2/Y_t$, originate from volatility arbitrage and noise trader arbitrage, respectively.

In the following two subsections we derive Π^{Vol} and Π^{NT}.

2.1 Volatility Arbitrage

Assuming Geometric Brownian Motion (GBM) in the true price P^* and a zero fee for the AMM, Milionis, Moallemi, Roughgarden, and Zhang [2023b] derive a closed form solution for Π^{Vol} in the absence of noise traders.[4] The term Volatility Arbitrage intuitively represents the source of the arbitrage profit. The higher volatility is, the greater and more frequent arbitrage opportunities are, thus greater arbitrage profit. In this setting, Π^{Vol} is equal to the expected impermanent loss. The formula of

[2]We call it "marginal" price because it is the price offered by AMMs for an infinitesimally small trade. Therefore, the average trade price for an order is the weighted average of the marginal price.

[3]Intuitively, the true price may be thought of as a weighted average of prices across all platforms, where the weights are defined by the extent platforms contribute to price discovery. More formally, the true price is an unobserved common stochastic trend in all price series that follow a Brownian semimartingale process.

[4]They termed Π^{Vol} as "Loss-Versus-Rebalancing" (LVR), because arbitrageurs rebalance their portfolio the reserve of the liquidity pool to align it with the true price.

Π^{Vol} for a CPMM is derived as follows:

$$\Pi^{\text{Vol}}_{t,t+h} = \int_t^{t+h} \frac{Y_s}{4}\sigma_s^2 ds \simeq h\frac{Y_t}{4}\sigma_t^2,$$

see Milionis et al. [2023b] for a detailed proof. The approximation holds for a sufficiently small h.

A zero-fee right gives the slot holder a significant advantage in the arbitrage race. This setting is different from Milionis et al. [2023b], who consider a competitive market with multiple zero-fee agents. We arrive at the same expression for Π^{Vol} because the competitive structure is not needed for a monopoly slot holder, who is motivated to eliminate price gaps continuously.[5]

It is profitable for a zero-fee arbitrageur to immediately align the marginal price with the true price without waiting for larger price deviations. Under the assumption of GBM, the probability of a price gap increasing is the same as it is for it decreasing. If the price gap increases to surpass the fee rate, non-zero fee arbitragers can take profit. Acting swiftly to align prices minimizes the chance of losing potential profits to competitors. This proves Π^{Vol} can be effectively employed even under limited competition.

2.2 Noise Trader Arbitrage

An important component of a market is the service it offers to market participants, who seek to adjust their portfolio, store/withdraw liquidity, exchange one currency or asset for another, for a variety of reasons. These are "convenience traders", but the literature commonly designates these as "noise traders" because their activities can induce noise in the "pricing errors". Unlike arbitrageurs, noise traders are motivated by aspects other than minimizing, $P - P^*$.

Including noise traders in the analysis is important for several reasons. First, noise traders influence the profit of arbitrageurs. A zero-fee arbitrageur will earn a profit from reverting the price discrepancies in $P - P^*$. Second, noise traders bring revenue to AMMs through fees, which supports liquidity provision[6].

Our analysis concentrates on the lower-bound of profits obtained from aligning the true price with the marginal price, deliberately excluding the impact of potentially predatory trading practices by zero-fee traders (see Conclusion for potential predatory practices). We assume a zero-fee arbitrageur reverts the marginal price to the true price immediately.

Let q_t be a binary variable, such that $q_t = 1$ if a noise trader arrives at time t. We assume that the arrival times, $\{q_t\}$ follow a Poisson point process with intensity λ. The distribution of the number of noise traders,

[5] In the context with noise traders (to be defined in the next section) the transactions of a monopoly slot holder need not be identical to the aggregate transactions of competitive zero-fee traders.

[6] This highlights another important aspect of designing AMMs. High trading fees may discourage noise traders from using the AMM, while lower trading fees should positively affect trading volume.

n, that arrive during the interval of time, $[t, t+h]$ is

$$\Pr\left(\sum_{s\in[t,t+h]} q_s = n\right) = \frac{(\lambda h)^n}{n!}e^{-\lambda h} \qquad \text{for} \quad n = 0, 1, \ldots.$$

When a noise trader arrives, they trade k_t units of the numéraire, Y, for Pk_t units of X. So, the total trading volume over the interval of time $[t, t+h]$ is given by

$$\int_t^{t+h} |k_s| dq_s,$$

and we allow k_t to have any symmetric distribution with zero mean and variance, $\text{var}(k_t) = \omega^2$.

A noise trader changes the reserve of numeráire in the liquidity pool from Y to Y', where $k = Y' - Y$, and the resulting price impact is given by the Constant Product Market Maker formula (CPMM),

$$P = \frac{Y}{X} \mapsto P' = \frac{Y+k}{XY/(Y+k)} = \frac{(Y+k)^2}{XY}.$$

From this it follows that

$$\frac{k}{Y} = 1 - \frac{Y'}{Y} = 1 - \sqrt{\frac{P'}{P}} = 1 - \exp\left(\frac{r'}{2}\right) \simeq -\frac{1}{2}r', \tag{2}$$

where $r' = \log\frac{P'}{P}$. This establishes the (logarithmic) price impact of a noise trade to be proportional to the relative trading volume, k/Y. Specifically,

$$r' = \log P' - \log P = -2\frac{k}{Y} + o\left(\frac{k}{Y}\right).$$

An arbitrageur earns profit by reversing the price imbalance in the pool. For example, if $P' < P^*$ then the arbitrageur can profit from buying X from the pool and selling it elsewhere at the higher price P^*. The total profit will depend on the price difference and the quantity that can be traded until the marginal price is aligned with P^*. The average purchase price will be between P' and P^*, as a result of "slippage". The arbitrageur's profit is given by

$$\Pi(P', P^*) = \int_{P'}^{P^*} (P^* - p)q(p)\mathrm{d}p,$$

where

$\int_{P'}^{P^*} q(p)\mathrm{d}p = X' - X^*$,

is the total quantity of X that the arbitrageur withdraws from pool. From the constant product rule, we can use this to infer that

$$q(p) = \frac{L}{2}p^{-3/2},$$

since $\int_{P'}^{P^*} q(p)\mathrm{d}p = X' - X^*$ equals

$$\frac{\sqrt{X'Y'}}{\sqrt{\frac{Y'}{X'}}} - \frac{\sqrt{X^*Y^*}}{\sqrt{\frac{Y^*}{X^*}}} = L\left(\frac{1}{\sqrt{P'}} - \frac{1}{\sqrt{P^*}}\right) = \int_{P'}^{P*} \frac{L}{2}p^{-3/2}\mathrm{d}p. \tag{3}$$

Using the expression for $q(p)$ we find that

$$\Pi(P', P^*) = \int_{P'}^{P^*} (P^* - p)q(p)\mathrm{d}p \tag{4}$$

$$= \frac{LP^*}{2} \int_{P'}^{P^*} p^{-3/2}\mathrm{d}p - \frac{L}{2} \int_{P'}^{P^*} p^{-1/2}\mathrm{d}p \tag{5}$$

$$= \frac{LP^*}{2} \left[-2p^{-1/2}\right]_{P'}^{P^*} - \frac{L}{2} \left[2p^{1/2}\right]_{P'}^{P^*} \tag{6}$$

$$= LP^* \left(-\frac{1}{\sqrt{P^*}} + \frac{1}{\sqrt{P'}}\right) - L\left(\sqrt{P^*} - \sqrt{P'}\right) \tag{7}$$

$$= L\left(-\sqrt{P^*} + \frac{P^*}{\sqrt{P'}} - \sqrt{P^*} + \sqrt{P'}\right) \tag{8}$$

$$= L\sqrt{P'}\left(1 - 2\sqrt{\frac{P^*}{P'}} + \frac{P^*}{P'}\right) \tag{9}$$

$$= Y'\left(1 - \sqrt{\frac{P'}{P^*}}\right)^2 = Y'\left(\frac{k}{Y^*}\right)^2 \tag{10}$$

$$\frac{k}{Y^*} = \frac{Y^* - Y'}{Y^*} = 1 - \frac{Y'}{Y^*} = 1 - \frac{L\sqrt{P'}}{L\sqrt{P^*}} = 1 - \sqrt{\frac{P'}{P^*}} \tag{11}$$

where the last equality follows from (2). The arbitrageur's profit from noise traders is random because the arrival of noise traders is random. But we are now ready to compute an arbitrageur's expected profit from noise traders. Over the interval of time, $(t, t+h]$, the expected profit from noise traders is given by:

$$\Pi_{t,t+h}^{NT} \equiv \mathbb{E}_t\left[\int_t^{t+h} Y_s'\left(\frac{k_s}{Y_s^*}\right)^2 dq_s\right] \tag{12}$$

$$= \mathbb{E}_t\left[\int_t^{t+h} (Y_s^* - k_s)\left(\frac{k_s}{Y_s^*}\right)^2 dq_s\right] \tag{13}$$

$$= \mathbb{E}_t\left[\int_t^{t+h} Y_s^*\left(\frac{k_s}{Y_s^*}\right)^2 dq_s\right] \tag{14}$$

$$= \int_t^{t+h} \mathbb{E}_t\left[\frac{k_s^2}{Y_s^*}\right] dq_s \tag{15}$$

$$= \int_t^{t+h} \mathbb{E}_t[k_s^2]\mathbb{E}_t\left[\frac{1}{Y_s^*}\right] dq_s \tag{16}$$

$$= \frac{1}{Y_t^*}\omega^2 \int_t^{t+h} \exp\left(\frac{1}{8}\int_t^s \mathbb{E}_t\left[\sigma_u^2\right] du\right) dq_s \tag{17}$$

$$\approx \frac{1}{Y_t^*}\omega^2 \int_t^{t+h} \exp\left[\frac{1}{8}\sigma_t^2(s-t)\right] dq_s \tag{18}$$

$$\approx \frac{1}{Y_t^*}\omega^2 h\lambda \frac{\exp\left(\frac{\sigma_t^2 h}{8}\right) - 1}{\sigma_t^2 h/8} \tag{19}$$

$$\approx \frac{1}{Y_t^*}\omega^2 h\lambda \tag{20}$$

where (14) comes from the symmetric distribution of k_s with mean 0, (15) holds from Fubini's Theorem, (16) comes from the independence between k_s and P_s, (17) comes from $Y_t = L\sqrt{P_t}$ and SDE of $\sqrt{P_t}$, and the constant variance of noise traders' volume assumption. The approximations (18), (19), and (20) holds for sufficiently small h.

3 Equilibrium Slot Price

In the previous Section, we derive the expected profit for a monopoly zero-fee arbitrageur. In this section we take into account the strategic considerations that arise in the bidding for the slot. Insight about the equilibrium price is gained by considering a risk-neutral and rational bidder for the slot. If the owner is guaranteed to hold the slot until expiration, 24 hours later, then the maximum willingness-to-pay is the expected profit from owning the slot. This defines an upper bound on the slot price.

However, the expected profit of a slot holder is different from the expected profit of a zero-fee arbitrageur, because the slot holder can be outbid prior to the expiration time. In Section 2 we demontstrate that the expected profit is positively related to volatility and the intensity of noise trader arrivals. If there is an increase in either of these variables, then the value of the slot increases. In these situations, it could become profitable for someone else to outbid the current holder. On the other hand, if the value of the slot decreases, then the current slot owner is stuck with the (now less profitable) slot until its expiration. This amounts to the slot having a short position on a call option that caps the profit of owning the slot.

At the same time, the slot mechanism discourages bids far below the true value of the slot. A low purchase price, B, increases the probability of being outbid before slot expiration, and the arbitrage that can be earned from the slot decreases as the likelihood of being outbid increases.

To determine the optimal bidding strategy for the auction participants, we first describe the proposed continuous auction. Next we assume symmetric perfect information to present the most simplified model for bidders. [7] Finally, we propose methodologies for a numerical solution for an optimal bid using a Q-learning algorithm.

3.1 The Proposed Continuous Auction

The Slot Mechanism for the AMM on the XRP Ledger (XRPL) allows the slot owner to trade with zero fees for 24 hours. The slot owner is

[7] The arbitrage profit in this system is a function of volatility, which is not directly observable but rather estimated. Therefore, a common value auction with imperfect information would be more aligned with real-world scenarios. In such a case, even if participants each have unbiased estimates, the variance in their volatility estimates could lead to bidding below the actual common value to avoid the winner's curse. See Krishna [2009] for detailed discussion about common value auction under various environments. The larger the variance in these volatility estimates, the greater the gap between the bidding and the common value is likely to be. This issue can be mitigated if options, volatility swaps, or products like the VIX are traded in the market. These instruments provide a publicly observable measure of volatility, offering a common reference point for all participants.

determined by an auction which has the following rules:

1. Slot ownership can be in one of three states:
   (a) Empty: No account owns the slot.
   (b) Occupied: An account owns the slot and there is more than 72 minutes until expiration.
   (c) Tailing: An account owns the slot and there is less than 72 minutes until expiration.
2. We let t denote time, where a unit of time corresponds to 72 minute intervals over 24 hours. The age of ownership is represented by the variable a, where $a = 0$ at the time ownership is acquired, such that slot expires once $a = 1$.[8] The Slot Mechanism defines a *discretized age* variable,[9]
$$s = s(a) = \lceil 20a \rceil / 20,$$
   that divides the age of ownership into twenty sub-intervals, such that $s = 0.05$ for $a \in (0, 0.05]$ (the first 72 minutes), $s = 0.10$ the next 72 minutes, and so forth. In the Empty state, with no ownership, we use the convention $s = a = 1$.
3. Accounts can place a bid at any time provided the bid is greater than, or equal to, the prevailing minimum bid. The minimum bid depends on the age of current ownership, a, and the last purchase price, B.

   The minimum bid is given by
$$\mathrm{MB}(B, a) = 1.05 \times B \times (1 - s^{60}) + M, \tag{21}$$
   where B is the most recent purchase price and M is the minimum slot price, which is defined by
$$M = \mathrm{LPTokens} \times \frac{\mathrm{TradingFee}}{25}.$$
   When ownership changes before the time of expiration ($a < 1$), the current owner is refunded
$$R(B, a) = B \times (1 - s(a)).$$

3.2 Continuous Common Value Auction

To determine the equilibrium of the continuous auction, we must determine the optimal bidding strategy. The problem is similar to solving for the optimal bid in a static auction with a minimum bid, however the possibility of being outbid prior to the expiration makes the continuous auction different.

[8]The variable a is identical to t using in Section 4.1.1. of "Automated Market Maker on XRP Ledger". We use t to denote continuous time below.

[9]$\lceil x \rceil$ is the ceil function that rounds x up to the nearest integer.

3.2.1 Insight from a Simplified Problem

To gain intuition of the problem we begin by considering a simplified version:

- There is perfect information and homogeneous bidders,such that all bidders assign the same value, V to slot ownership.
- Outbidding is only possible immediately after a successful bid, which we label instantaneous outbidding.
- If instantaneous outbidding occurs, the slot holder will suffer a 5% loss, because the refund rate is 95% immediately after a successful bid.
- If instantaneous outbidding does not occur, then the slot will be in the possession of the holder for the full duration (24 hours).

In this simplified situation, a successful bidder will earn the profit when

$$\Pi \equiv \mathbb{E}_t \left[\int_t^{t+1} \pi_a \mathrm{d}a \right].$$

Now we assess a new bidder buying the slot in this simplified version. Let B denote the most recent purchase price. A successful new bid, B', for the slot must meet two basic requirements. First, it must meet the the minimum bid requirement, which is $B' \geq 1.05B(1 - s^{60}) + M$. Second, it must be sufficiently large to deter competitors from instantaneous outbidding. An instantaneous outbid is profitable if $\Pi \geq 1.05B' + M$. A new success bid is defined as

$$B' \geq \frac{\Pi - M}{1.05} \equiv B_L,$$

which is the threshold at which instantaneous outbidding becomes unprofitable. For example, if $\Pi = 110$ and $M = 5$, then the bid must be at least $B_L = 100$ to deter an instantaneous outbid.

This simplified bidding problem is illustrated in Figure 1. The expected profit of a slot holder is illustrated by the shaded areas in the figure. A bid below B_L will not deter an immediate outbid. The partial refund is illustrated with the solid blue line. We can see that the partial refund always induces a negative profit, which is represented by the height of the red shaded area. A bid above B_L will deter an immediate outbid making the expected profit the difference between Π and the 45 degree line, whish is illustrated as the blue shaded region. Naturally, a bid that is larger than Π will result in an expected loss.

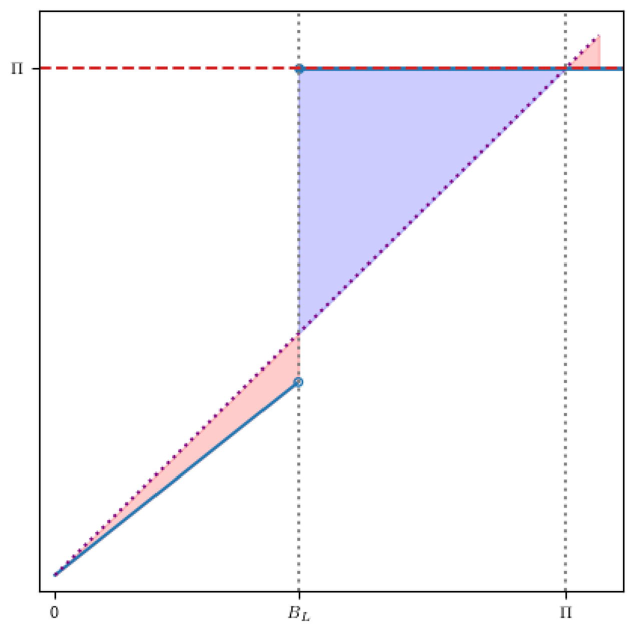

Figure 1: The expected profit of a slot holder in a simplified problem.

In this simplified version of the problem the optimal bidding strategy is

$$B^* = \max(B_L, M),$$

provided that the minimum slot price makes it profitable to bid, i.e. $M \leq \Pi$.

The actual problem is more complicated for two reasons. First, deterring instantaneous outbidding need not preclude outbidding at a later point in time before slot expiration. Second, the value process is a stochastic process that depends on volatility, liquidity pool size, and the intensity of noise trader arrivals. So, the actual problem is a dynamic stochastic optimization problem, where outbids occur at random times and the the duration of ownership will be a random variable.

3.2.2 The Dynamic Problem

A key variable in the dynamic problem is the random duration of slot ownership,

$$\delta(B) \in [0, 1],$$

which is defined as the time where it becomes profitable to outbid the current slot holder. This type of random variable is called a stopping time. The expected duration is increasing in B, which is important for determining the optimal bidding strategy.

The following expression is for the expected value of winning the slot at time t with the bid B,

$$V_t(B) \equiv \mathbb{E}_t\left[\int_t^{t+\delta(B)\wedge 1} \pi(Y_a, \sigma_a^2)\mathrm{d}a + R(B, \delta(B))\right], \tag{22}$$

which is the profit earned until outbidding occurs, and the refund is distributed.

Outbidding is profitable at time $t' = t + d$ if there exists a bid, B', that is both profitable and meets the minimum bidding requirement, i.e. $\min(B', V_{t+d}(B')) \geq \mathrm{MB}(B, d)$. Since $\delta(B)$ is the first time outbidding is profitable, we can infer that

$$\delta(B) = \min\left\{d : \mathrm{MB}(B, d) \leq \min(B', V_{t+d}(B')) \text{ for some } B'\right\}, \tag{23}$$

with the convention that $\delta(B) = 1$ if no solution exists for $d \leq 1$.

From (22) it is clear that the value function, $V_t(B)$, depends on the slot duration. More precisely, it depends on the conditional distribution of an outbid over the period $(t, t+1]$, which is characterized by the distribution of $\delta(B)$ given the information available at time t. Similarly, from 23 it is evident that the slot duration, $\delta(B)$, depends on the future path of the value function, $V_{t+d}(B)$, which is also random, because it depends on the volatility process, $\{\sigma_t\}$, the intensity of noise trader arrivals, as well as Y_t. Once the random properties of these variables are fully specified the value function can be obtained by solving the recursive dynamic optimization problem. One way to solve this problem is by the methods of reinforcement learning, and we include the pseudo code for this in the Appendix.

While the solution to the problem will be context specific, we can characterize some of its generic properties in relation to the simplified problem we consider above. The threat of an instantaneous outbid will impose a lower bound, B_L, on bids that will be profitable. This lower bound will increase, because the bid has to be sufficiently large to both deter instantaneous outbidding, as well as decrease the likelihood of outbidding at at later point in time. The lower bound is therefore given by

$$\mathrm{MB}(B_L, 0) = V_t(\mathrm{MB}(B_L, 0)). \tag{24}$$

This solution exists assuming V_t is continuous with respect to B. The proof is simple. For sufficiently large B, no one will outbid until duration. Therefore, $\lim_{B\to\infty} V_t(B) = \mathbb{E}_t\left[\int_0^1 \pi_a \mathrm{d}a\right]$. If $V_t(B) < B$ for all B in equilibrium, no one will outbid for the slot, so the bidder can take the slot for the full duration with a small bid obtaining positive profit. Therefore, there exists B_0 such that $V_t(B_1) > B_1$. By continuity of $V_t(B)$, we can find $B^* \in [B_1, \infty)$ such $V_t(B^*) = B^*$.

Note that $\mathrm{MB}(B_L, 0)$ is on the 45 degree line, where the bid is identical to expected profit. Meanwhile, it is not profitable to bid a larger amount than

$$B_U \equiv V_t(\mathrm{MB}(B_L, 0)),$$

The optimal bid, B_*, is therefore pinned down between B_L and B_U. An illustration of what the solution to the dynamic problem could look like is in Figure 2.

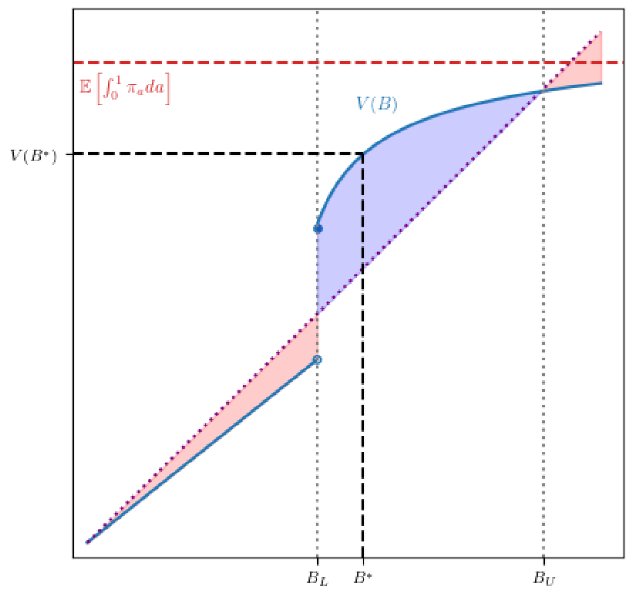

Figure 2: Illustration of value function, $V(B)$, in stochastic dynamic optimization problem.

4 Do LPs benefit from Auction Slot Mechanism

The Auction Slot Mechanism is beneficial to LPs if they earn more profit than in an identical AMM without the slot. Specifically, the value of the amount of burned LP shares from the auction plus fees from non-slot holders must be greater than the total amount of fees in an identical AMM pool which does not have the slot. Logically we show the aggregate of fees contributed by arbitrageurs in a system without the slot would inevitably be lower than the profits of a zero-fee arbitrageur. Therefore, if the the profits of the zero-fee arbitrageur are higher and if the auction can sufficiently burn enough of that value then the Auction Slot Mechanism will benefit LPs. We start by exploring the amount of fees each system generates.

Assuming no noise traders, Milionis, Moallemi, and Roughgarden [2023a] derive a closed-form solution for fees collected from arbitrageurs in a system with no slot and find that the collected fees are smaller than Π^{Vol}, the volatility profit of arbitrageurs. Despite a lack of research on fee collection in the presence of noise traders, it is reasonable to posit that in a system without a slot mechanism, the collected fees are sometimes not sufficient to compensate Impermanent Loss of LPs, as reported in Kim [2023]. On the other hand, the total arbitrage profit from zero-fee trader is greater than the impermanent loss. If the auction mechanism is appropriately designed, it can improve the profit of LPs. In this section, we formulate that expected total revenue from Auction Slot, and study the condition when it is maximized.

4.1 Expected Total Revenue from Auction Slot

The expected total revenue from the auction for a given time interval $[0, T)$ is

$$\mathbb{E}_0 \left[B_1^* - R(B_1^*, \delta_1) + B_2^* - R(B_2^*, \delta_2) + \dots \right] \tag{25}$$

where B_i^* is i-th effective bidding made at time t_i and δ_i is the duration of the i-th slot. By adding and subtracting $V_{t_i}(B_i^*)$, the equation (25) can be rewritten as:

$$\mathbb{E}_0 \left[\sum_{i=1}^{\infty} \{B_i^* - V_{t_i}(B_i^*)\} + \sum_{i=1}^{\infty} \{V_{t_i}(B_i^*) - R(B_i^*, \delta_i)\} \right] \tag{26}$$

Using the equation (22), we can rewrite the above equation:

$$\mathbb{E}_0 \left[\sum_{i=1}^{\infty} \{B_i^* - V_{t_i}(B_i^*)\} \right] + \mathbb{E}_0 \left[\int_0^T \pi_a \mathrm{d}a \right] \tag{27}$$

The second term is the expected profit of a zero-fee arbitrageur on the given interval. The first term is always non-positive, because the maximum willingness to pay a rational bidder is the value of the slot, therefore $B_i^* \le V_{t_i}(B_i^*)$ always holds.

The expected total revenue from auction slot is maximized when the sum of the loss is minimized. Figure 2 illustrates the loss from each bid, $B_{t_i}^* - V_{t_i}(B_{t_i}^*)$, and shed light on how to minimize the loss at the same time. When $B_U = B_L$, i.e., $MB(B_L, 0) = B_L$, $B_{t_i}^* - V_{t_i}(B_{t_i}^*) = 0$ holds. However, the current MB function does not satisfy this condition. We can consider using a Minimum Bid function MB_1 which is similar to the current $MB(B, a)$, such that

$$MB_1(B, a) = \left\{1 - (1-a)^{60}\right\} \left[1.05 \cdot B \cdot (1 - s^{60}) + M\right] + (1-a)^{60} B.$$

$MB_1(B, a)$ satisfies the condition $MB(B_L, 0) = B_L$ and has a similar shape to the original one for sufficiently large a.

To summarize, under the current mechanism, slot holders have the opportunity to retain a portion of their arbitrage profits without committing it entirely to bids. However, by designing appropriate auction mechanism which is compatible with blockchain consensus algorithm, LPs can extract the entire arbitrage profit through slot auction.

5 Conclusion

Ripple's proposed Auction Slot Mechanism has several potential affordances for AMM designs. Specifically, it may create a positive feedback loop in the AMM system that increases profits for both the slot holder and LPs. Doing so would create deeper and more stable liquidity in the entire system.

We prove mathematically that the slot holder generates more profit from the slot auction than an AMM system without the slot. Zero-fee arbitrageurs are capable of maintaining more efficient pricing in the AMM compared to scenarios where fees are present. They actively align price discrepancies with other markets, fostering a transfer effect in market liquidity. This positively effects noise traders by giving them deeper and more stable liquidity. [10] This provides an environment for greater trading activity in the liquidity pool. On top of that, we also prove that the profit for LP is dependent on the Auction Slot Mechanism, and suggest an improvement to increase the profit of LPs.

However, the discussion in this paper has the following limitation. First, we assume the perfect information. However, the assumption does not hold in reality, because the true price, spot volatility, and their stochastic processes are not directly observed rather estimated. The variation in the estimates among bidders could lead to bidding below the true common value to avoid winner's curse; see Chapter 5 in Krishna [2009] for more detailed discussion. Second, it should be noted that even for zero-fee arbitrageurs, significant investment in a system capable of high-frequency arbitrage is required, implying substantial fixed costs. In such a scenario, bidders with greater financial resources could potentially monopolize the arbitrage market. For instance, a well-funded bidder might consistently place bids higher than the actual arbitrage profit to secure the slot for themselves. This strategy could eventually force other bidders, unable to sustain the fixed costs, out of the market. Once competitors are eliminated, the dominant bidder can secure the slot with bids significantly lower than the expected arbitrage profit.

Finally, the issue of front-running in DEXs can be exacerbated in blockchain like Ripple Network, where network fees are very low, and randomized transaction ordering is employed. For instance, according to the XRP Ledger's documentation, a zero-fee arbitrageur can operate up to five accounts with zero fees. Ripple Network uses a pseudo-random transaction ordering algorithm, which calculates an account key for each transaction, combining the account ID and the salt using XOR operation, and the transactions are sorted based on the account key within a block. However, the strategic generation of multiple accounts can help to predict the intervals of these priorities. Utilizing five accounts can significantly increase the likelihood (upto approximately 83.3% change) of successfully executing a sandwich attack. See Tumas, Pontiveros, Torres, and State [2023] for strategic account generation for front-running. The profitability of these manipulative trading strategies, however, can vary with the available order options, such as slippage limit or exeucte-only when an order ranks higher in priority than those of zero-fee arbitrageur, and the noise traders' adaptation to transaction costs.

[10] If noise traders are able to strategically split and place their orders, this also leads to larger volume orders at a reduced effective trading cost

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A Pseudo Code for the Value function Estimation

The problem of optimal bidding in the continuous auction can be represented by the 4-tuple

$$\langle (\sigma_t, Y_t),\ \text{bid},\ \text{stochastic processes of } (\sigma_t, Y_t),\ \pi_t \rangle\,.$$

This aligns with the standard Markov Decision Process(MDP) framework of State(S), Action(A), Transition Probability(P), and Reward(R). We utilize Monte Carlo method in Reinforcement Learning(RL) to solve this MDP. A detailed discussion of Monte Carlo method in the context of RL can be found in Chapter 5 of Sutton and Barto [2018]. For notational convenience, we represent the state at time t as $S_t = (\sigma_t, Y_t)$.

As discussed in the in Section 3, the auction's outcome is influenced by one's own and other participants' bidding strategies, characterizing it as a non-cooperative multi-agent game. Under the assumptions of perfect information and symmetric agents, we solved the optimal bidding problem given unknown $V_t(B)$.[11] We approximate V_t in discrete time framework by

$$V^{(\theta)}(S_t, B) \equiv V_t^{(\theta)}(B) \approx V_t(B)$$

using a Deep Neural Network with parameters θ, focusing on the cases where the slot auction results are settled only at K discrete times $t \in \{t_k\}_{k=0}^{K-1}$ for $t_k = \frac{k}{K}$.

Let $\mathbb{P}_{t_i}$ denote a probability measure for a set of events conditional on all variables observed at time t_i and $\mathbb{1}_D(x)$ be an indicator function which yields 1 if $x \in D$, and 0 otherwise. Then, the following equation holds for the value function:

$$\begin{aligned} V_{t_0}^{(\theta)}(B) = \sum_{k=0}^{K-1} \Bigg\{ & \mathbb{E}_{t_0}\left[\Pi^{\text{Arb}}_{t_k, t_{k+1}} \middle| \mathbb{1}_{\{\cap_{j=0}^{k} D_{t_j}(B)\}}(\mu) = 1 \right] \mathbb{P}_{t_0}\left(\mathbb{1}_{\{\cap_{j=0}^{k} D_{t_j}(B)\}}(\mu) \right) \\ & + \mathbb{E}_{t_0}\left[R(B, t_k) \middle| \mathbb{1}_{\{\cap_{j=0}^{k-1} D_{t_j}(B) \cap D^c_{t_k}(B)\}}(\mu) = 1 \right] \mathbb{P}_{t_0}\left(\mathbb{1}_{\{\cap_{j=0}^{k-1} D_{t_j}(B) \cap D^c_{t_k}(B)\}}(\mu) = 1 \right) \Bigg\} \end{aligned} \tag{A.1}$$

for $D_{t_j}(B) = \{ S_{t_j} | MB(B, t_j) \leq B_{U,t_j} \}$ with $B_{U,t_j} = V_{t_j}^{(\theta)}(B_{U,t_j})$. Here, $D_{t_j}(B)$ denotes the set of events where profitable outbidding is feasible, as depicted in Figure 2. For a given trajectory $\mu = (S_{t_0}, S_{t_1}, \ldots, S_{t_{K-1}})$, the equation $\mathbb{1}_{\{\cap_{j=0}^{i} D_{t_j}(B)\}}(\mu) = 1$ holds when profitable outbidding has been unfeasible from t_0 to t_i.

The application of Deep Neural Networks in finance necessitates careful consideration due to their inherently opaque nature. To foster transparency and maintain alignment with the equilibrium we derived in Section 3.2.2, we propose a specific formulation of $V^{(\theta)}(S, B)$ in accordance

[11] The value function V_t in our context is dependent on the bid B, making it an action-value function. In RL, the action-value function is typically denoted as Q, while the state-value functions are represented by V. On the other hand, in the field of Auction Theory, the notation V is conventionally used to signify the value gained from an auction. In our work, we adhere to the notation practices of Auction Theory, using V to denote our action-value function, despite the deviation from the standard RL terminology.

with the insights illustrated in Figure 2:

$$V^{(\theta)}(S_t, B) = \begin{cases} 0.95B & \text{if } B < B_L^{(\theta_2)}(S_t) \\ f^{(\theta_1)}(S_t, B) \cdot \pi(S_t) & \text{Otherwise} \end{cases} \tag{A.2}$$

where $\pi(S_t) \approx \mathbb{E}_t\left[\int_t^{t+1} \pi(S_a)\mathrm{d}a\right]$ represent the expected arbitrage profit over the maximum duration, the function $f^{(\theta_1)}(S_t, B)$ in the range $(0, 1)$ is a neural network with a sigmoid output activation function. This formulation ensures $V^{(\theta)}(S_t, B) < \mathbb{E}_t\left[\int_t^{t+1} \pi(S_a)\mathrm{d}a\right]$. Additionally, as we demonstrated in equation 24, $MB(B_L, 0) = B_U = V_t(B_U)$ holds. Direct numerical calculation of B_U for each S_t satisfying $V^{(\theta)}(S_t, B_U) = B_U$ would require separate optimization processes whenever S_t is encountered in the simulation, leading to a significant computational load. To circumvent this, we introduce the neural network $g^{(\theta_2)}(S_t)$ to estimate B_U so that $V^{(\theta)}(S_t, g^{(\theta_2)}(S_t)) = g^{(\theta_2)}(S_t)$, thereby improving the computational efficiency of the model.

The pseudocode for estimating $V^{(\theta)}$ using Monte Carlo method is as follows:

Algorithm 1 Q-learning for estimating $V^{(\theta)} \approx V$

Hyperparameters: Step size $\alpha \in (0, 1]$, termination criteria, the number of trajectories N, and discreteness parameters K.
Initialization $\theta = (\theta_1, \theta_2)$ for $V^{(\theta)}(S, B)$
loop
    Generate $S_{j,t_0} = (\sigma_{j,t_0}, Y_{j,t_0})$ from an arbitrary initial distribution for $j = 1, \ldots, N$.
    Generate $\{S_{j,t_k}\}$ for $j = 1, \ldots, N$ and $k = 1, \ldots, K-1$ based on the stochastic processes of the state variables.
    Calculate Loss $L^{(\theta_2)}$ and update θ_2

$$L^{(\theta_2)} = \frac{1}{NK} \sum_{j=1}^{N} \sum_{k=0}^{K-1} \left(g^{(\theta_2)}(S_{j,t_k}) - V^{(\theta)}(S_{j,t_k}, g^{(\theta_2)}(S_{j,t_k})) \right)^2$$

    Generate $b_j \sim U[\Pi^{(\theta_3)}(S_{j,t_0})/3, \Pi^{(\theta_3)}(S_{j,t_0})]$ for $j = 1, \ldots, N$
    Calculate the target $y_j^{(\theta_1)}$, loss $L^{(\theta_1)}$, and update θ_1

$$y_j^{(\theta_1)} = \sum_{k=0}^{K-1} \Pi_{t_k, t_{k+1}}^{Arb}(S_{j,t_k}) \mathbb{1}_{\{\cap_{j=0}^{k} D_{t_j}(b_j)\}}(\mu_j) + R(B, t_k) \cdot \mathbb{1}_{\{\cap_{j=0}^{k-1} D_{t_j}(B) \cap D_{t_k}^c(B)\}}(\mu_j)$$

$$L^{(\theta_1)} = \left(V^{(\theta)}(S_{j,t_0}, b_j) - y_j^{(\theta_1)} \right)^2$$

end loop If termination criteria is met