Optimal Staking in a DAO: A Partial Equilibrium Analysis

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Abstract

This paper investigates an optimal staking decision problem of an agent who participates in a DAO. We set up the model that highlights the tradeoff between the staking reward and illiqiudity from staking tokens. The model allows fully analytic solutions. The model provides various implications. The agent infrequently changes the staked token position. The frequency increases with the staking reward and decreases with the cost of unlocking. The liquid token holdings can be used to hedge the bad shock against the staked position particularly when wealth is low. The agent increases the total token holdings as wealth increases, which means taking more risk as wealth increases. However, the staking ratio, i.e., the proportion of the staked tokens out of the total token holdings can increase or decrease with wealth depending on the types of investors.

Keywords: Blockchain, Proof of Stake, Optimal Staking, DAO (Decentralized Autonomous Organization), DeFi (Decentralized Finance).

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1. Introduction

Recently, more and more blockchain protocols use Proof of Stake (POS) rather than Proof of Work (POW) for the underlying consensus mechanism for block validation. The increasing criticism on the fossil energy consumption by using POW has also contributed to this trend. The POS mechanism requires block validators to stake tokens into the protocol. As many Defi (Decentralized Finance) applications, yield farming protocols, CEXs (Centralized Exchanges) and DEXs (Eecentralized Exchanges) provide stake pool services, nowadays individual investors easily stake their tokens through these service providers in order to earn additional returns. In addition to the popularity of general DeFi (Decentralized Finance) protocols, there has been a boom of DAOs (Decentralized Autonomous Organizations) as a form investment club. Unlike the DeFi protocols these DAOs target not only on-chain investment but also off-chain investment. Regardless whether DAOs use POW or POS for basic block validation mechanism, they use governance tokens for voting for investment decision making (see Section 2.2 for the detail). This process also requires staking governance tokens when making decisions and when distributing the returns.

Note that while the staked tokens generate the staking reward, they are illiquid because there exists the cost from unlocking tokens due to the locking period and fees. This tradeoff exists regardless of whether investors stake tokens in CEXes, DEXes, DeFi protocols, or DAOs like investment clubs. Therefore, investors need to optimally choose between holding tokens in a liquid account and staking tokens in a locked account. The aim of this paper is to formalize the agent's problem of staking tokens and to understand the agent's optimal trading behaviors.

We extend a standard model of the agent's consumption and portfolio selection problem into the case where the agent additionally has an option to stake tokens into a DAO. Then, the agent's wealth consists of the three components: cash, liquid token holdings, and staked token holdings. There is a cost from unlocking the staked tokens while the staked tokens generate the staking reward. Following the standard utility platform literature (Cong et al. (2019, 2021b), Cong et al. (2021a)), we assume that the agent can have utility not only from consumption but also from staking tokens. Based on this utility setup we can categorize broadly two types of heterogeneous investors: active participants and general investors. Active participants of the DAO represent those who have a relatively higher weight on utility from staking than on the weight on utility from consumption. For example, active participants include the initial founders, the DAO's team members, early participants, and blockchain developers. General investors are the rest who have a higher weight on utility from consumption, including the agents who have no utility from staking tokens.

While the agent's problem has two singular controls, which leads to complicated variational inequalities, we can obtain fully analytic solutions by using a duality principle. We find that there exists an inaction interval at which the agent does not change the staked position (i.e., does not stake more tokens nor unlock the staked tokens). Whenever the wealth hits a certain threshold level, the agent stakes additional tokens. Whenever wealth hits the zero boundary, the agent have two different behaviors depending on the size of cost of unlock the staked tokens. If the cost is high, it is never optimal to unlock the staked tokens. Instead, the agent makes a negative liquid token position to offset the staked token position so that total wealth never reaches the zero.[1] If the cost is low, the agent optimally liquidates staked tokens to avoid wealth from reaching zero.

The model provides several important predictions of the optimal staking behavior. First, we find that the frequency of changing the staked token position is higher as the cost decreases or as the reward increases. The reason regarding the cost is straightforward: a low cost can enable the agent to change the position frequently. Regarding the staking reward, if the reward increases, there is more incentive to stake tokens. Thus, the agent increases the amount of staked tokens in response to a small increase in wealth.

[1]Note that the shortsale of crytocurrecy tokens are available in many cryptocurrency exchanges by using futures contracts.

Second, what about the optimal investment and consumption strategy within the inaction interval? To make it clear, consider a simple case in which the token price continuously increases (or decreases) so that wealth increases (or decreases). In this case, given that the agent does not increase (or liquidate) the staked tokens yet, what is the optimal choice of the agent in terms of allocating liquid tokens and consumption? As the token price increases in the inaction region, the optimal decision is to increase both consumption and liquid token holdings. However, the ratio of consumption to wealth decreases with wealth, which implies that the liquid token holdings increase much more than the increase in consumption. In other words, the agent takes more risk as wealth increases within the inaction interval. There are two forces to derive the result of increasing risk-taking behavior: (i) as wealth increases, the borrowing constraint is more relaxed. (ii) by increasing liquid token investment, the possibility of reaching out the boundary increases so that it can effectively increase the reward in the future.

Third and perhaps the most important, we find that the effect of the staking reward is fairly different depending on the type of investors. For general investors, the increase in the staking reward leads to the increase in the staking ratio (i.e., the proportion of staked tokens of the total token holdings). However, for active participants who have more attachment to the DAO, the increase in the staking reward decreases the staking ratio. To understand this result, note that staking provides the addition income stream to the agent, which makes the problem comparable to a standard consumption-investment problem with *stochastic income*. The increase in the staking reward effectively increases the total wealth (that is, current wealth and the present value of income). Therefore, the total token holdings increase with the reward, which is true for any types of investors. The difference is that out of the increase in the total token holdings, the increase in the staked token is greater than the increase in the liquid token holdings for general investors, while the opposite is the case for the DAO's active participants. Note that as the staking reward increases, the marginal utility from consumption of the active participants become higher (given that they have high staking due to high utility from staking), which implies that the risk-averse active participants have higher consumption smoothing motives and thus they need to increase liquid token holdings as a buffer against bad shocks.

Our paper are related to the POS literature such as Cong et al. (2022), John et al. (2022), Rosu and Saleh (2021), and Saleh (2021). Among these papers, the closest one is Cong et al. (2022). Our paper investigates the optimal staking behavior under illiquidity and inconvenience generated from the existence of a locking period and gas fee while there is no such friction in Cong et al. (2022) and instead, their focus is to determine the staking reward in equilibrium. In addition, our paper models general aspects of DAOs specializing both on-chain and off-chain investment.

The rest of the paper proceeds as follows. Section 2 provides the model. The solution analysis and the optimal policies are presented in Section 3. Section 4 investigates various implications of the optimal policies. Section 5 provides concluding remarks.

2. Model

We consider a DAO platform run under the POS consensus mechanism and set up an agent's problem who optimally consumes and invests in the DAO. We first describe the mathematical setup of the model in Section 2.1. Then, in Section 2.2 we will provide the detailed background of why we lay out the model in such a way by introducing various DAOs currently operated on blockchain. This explanation includes how to interpret key parameters of the model.

2.1. Setup

The DAO issues a cryptocurrency token. Assume that the dynamics of the token is given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

where B_t is a standard Brownian motion in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, μ is the expected growth rate of the token price, and σ is its volatility. μ and σ are positive constants. We will provide discussion for the token usages or rights regarding the characteristics of the DAO in Section 2.2.

At each time t the agent decides how much to consume and how much to allocate the fund between the tokens and the risk-free asset providing a constant rate of return $r > 0$. In addition to holding tokens, the agent can stake some of the tokens on the blockchain of the platform.[2] Let x_t be the number of tokens that are held by the agent (not being staked) and let k_t be the number of tokens that are staked at time t. Then, the agent's total token holdings are $\pi_t := x_t + k_t$.

The dynamics of k_t is given by

$$dk_t = -\delta k_t dt + d\mathcal{G}_t^+ - d\mathcal{G}_t^-, \tag{1}$$

where $\mathcal{G}_t^+$ (or $\mathcal{G}_t^-$) is the cumulative amount of tokens being staked (unlocked) at t and $\delta > 0$ represents the depreciation rate of the staked tokens by slashing or burning or the management cost of the DAO.

The agent earns rewards over time proportional to its staking amount. That is, the instantaneous staking reward is $\phi k_t S_t dt$, where ϕ is constant.[3] On the other hand, there is a cost from unlocking the staked tokens, $\rho S_t d\mathcal{G}_t^-$, where the constant $\rho \in [0, 1)$ is the token

[2] We can also think that the agent can stake the tokens through staking service providers such as StakeWise, StakeWithUs, Stakin, Staking Facilities, Stake.fish and so on.

[3] We discuss the staking reward in Section 2.2 in detail.

deduction rate per one unit of token withdrawal. Without loss of generality we assume that there is no cost when locking up tokens for staking.[4]

Let W_t be the agent's wealth at t. W_t consists of three components: liquid token holdings, risk-free assets, and staked (or illiquid) token holdings. Then, theses three components determine the wealth dynamics dW_t as follows with an initial with $W_0 = w_0$:

$$\begin{aligned} dW_t &= \pi_t dS_t + r(W_t - \pi_t S_t)dt + \phi k_t S_t dt - S_t d\mathcal{G}_t^+ + S_t(1-\rho)d\mathcal{G}_t^- - c_t dt \\ &= [rW_t + (\mu - r)\pi_t S_t + \phi k_t S_t - c_t]dt - S_t d\mathcal{G}_t^+ + (1-\rho)S_t d\mathcal{G}_t^- + \sigma \pi_t S_t dB_t, \end{aligned} \tag{2}$$

where $\pi_t dS_t$ is the instantaneous change of the value of the total token holdings, $r(W_t - \pi_t S_t)dt$ is that of the risk-free asset holdings, $\phi k_t S_t dt$ is the reward from the staked token holdings, $\rho S_t d\mathcal{G}_t^-$ is the cost from unlocking tokens, and $c_t dt$ is instantaneous consumption.

Finally, the agent's problem is stated as follows. Given the initial wealth $W_0 = w$, the initial staking $k_0 = k$, and the initial token price $S_0 = s$, the agent maximizes his/her expected utility by choosing consumption (c_t), liquid token holdings (x_t), and staking decisions ($\mathcal{G}_t^+$ and $\mathcal{G}_t^-$). More precisely, for $\omega \in (0, 1]$,

$$V(w, k, s) := \max_{(c_t, x_t, \mathcal{G}_t^+, \mathcal{G}_t^-)} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left(\omega u_1(c_t) + (1-\omega)u_2(k_t S_t)\right)dt\right] \tag{3}$$

with $(W_0, k_0, S_0) = (w_0, k, s)$ subject to the budget constraints (2) and the borrowing constraint:

$$W_t \geq 0 \quad \text{for all} \quad t \geq 0, \tag{4}$$

where $\beta > 0$ is the subjective discount factor and ω is the weights between utilities from

[4] We could assume that there is also a cost when locking up tokens. All the results are preserved if the cost from unlocking is higher that when locking.

consumption and staking. In our model, we suppose $u_1(z) = u_2(z) = u(z)$ with

$$u(z) = \frac{z^{1-\gamma}}{1-\gamma}, \quad \gamma > 0,\ \gamma \neq 1.$$

Note that the agent with $w < 1$ has a utility $u(k_t S_t)$. This can be considered as the utility gains from transacting at the platform as usually assumed by utility token platform models (e.g., Cong et al. (2019, 2021b) and Sockin and Xiong (2022)). In the case of a DAO, we can alternatively assume that the agent also obtains utility from making investment and governance decisions for the DAO as a stake holder (owner). Therefore, utility increases with the value of staking, $k_t S_t$.

2.2. Background: How DAOs Work

In this section, we provide more discussion on our modelling approach. The interpretation of the model depends on the DAO's objective, structure, and operation mechanism, which are specified by the smart contracts on the blockchain. More specifically, the sign and the size of key parameters determine the characteristics of the platform. In order to understand why and how investors stake tokens in a DAO, we further explain the types of DAOs currently being operated.

First, a blockchain platform run under the POS mechanism requires block validators comparable to miners in the POW mechanicsm. Block validators stake tokens for receiving the rewards in addition to monetary benefits from the token price increase. The reward consists of two components: block rewards and gas fees paid by customers when they execute smart contracts for various purposes. In our model we do not distinguish the block rewards and the gas fees. Simply we assume that for each small time interval $[t, t + dt)$, total $\phi k_t dt$ number of tokens are given to the investor.

Second, for some DAOs (but not all DAOs), token stakers are entitled with the governance right that has a similar feature of the equity ownership right. Thus, the stakers in a DAO receive tokens as a reward generated by the project that the DAO executed. This type of DAO is similar to a conventional investment club, but the crucial difference between the two is that the DAO is run on blockchain and the investment decisions are made by voting by the token stakers. The very first example is the Ethereum DAO in 2015. Any staker can submit an investment proposal and the proposal is approved upon voting among stakers. In this case, the stakers receive returns from the investment. Since then, DAOs have mostly focused on on-chain investment such as NFT trading, other cryptocurrency investment on other cryptocurrency or ICOs (Initial Coin Offerings) in a DeFi (Decentralized Finance) world. However, recently there also came out DAOs that specialize off-chain investment. For example, the very first example of a DAO for off-chain investment is the Wyoming DAO for real estates in Wyoming in 2021. In sum, theses investment club-like DAOs are run for investing various assets including physical and virtual assets by using the idea of diversification and risk-sharing and by relying on the wisdom of the crowd. We assume that if the DAO size is large enough, it can generate constant flow of investment return ϕ.

With these two cases in mind, we provide some more detail comments regarding the interpretation of important model parameters:

- **Cost from unlocking the staked tokens**: $\rho > 0$ implies any cost generated by inconvenience, illiquidity, gas fee, and/or disincentive when unlocking the staked tokens. The inconvenience or illiquidity often is driven by the existence of the locking period. Note that for POS protocols with a long locking period, there often exists a market for staked tokens. The staked tokens are usually traded at discount.[5] In this case, ρ captures the difference between the liquid token price and the staked token price.

  Note that ρ could also represents the managerial or operational expenses for the DAO. The assumption is that the expenses are shared by stakers proportional to the share: the expense linearly increases with the number of tokens being staked. For example, it

[5] See, for example, `https://www.bitstamp.net/markets/eth2/eth/`

can be the cost from hiring specialists such as lawyers, accountants, or realtors if the DAO invests in real assets (e.g., Wyoming DAO). If the DAO is legally defined as an investment club, it is also subject to corporate income taxes.

- **Depreciation**: The depreciation rate of staked tokens, $\delta > 0$, have different meanings according to the nature of the DAO. POS platforms usually impose a slashing mechanism to disincentivize any misbehaving (dishonest) activities from the stakers. Slashing is required as a part of the consensus mechanism even if a staker do not have any intention of misbehaving. For example, for some protocols stakers' duty is to post the instantaneous prices of any tokens from different exchanges for each blcock validation time. If a posting is significantly deviated from the mean (or median) value of all the posting, the corresponding poster gets punished so that some of his/her staked tokens are slashed. In this case, this slashing continuously happens over time, which can make δ. Similarly burning of staked tokens happens for an algorithmic stablecoin platform during the times when the demand of the stablecoin increases. In this case, the tokens (of stakers) are burned as new stable tokens are issued in a rate of δ.

- **Staking reward**: $\phi > 0$ simply implies that the agent receives the reward by staking tokens in a POS staking pool or by participating in a investment club-like DAO as we describe above.

  Note that the staking reward $\phi k_t S_t dt$ directly enters the budget dynamics in (2). This implies that the reward is given as liquid tokens. For some staking pools, one cannot withdraw the reward tokens for a while (or until the lockup period ends). This case can be interpreted as the case in which the agent trades the staking reward for liquid tokens at the staked token market as soon as he/she receives the reward. In this case, modeling ϕ requires to take such a discount into account. In addition, there are decentralized applications for staking, yield farming applications, and cryptocurrency exchanges that provide staking services. Many investors use these service providers by paying service fees. In the case that investors use these services, ϕ is smaller than the actual reward rate provided by the direct on-chain staking. However, this consideration only has a quantitative impact, but does not change the qualitative result of the model.

3. Solution

In this section, we first derive a dual singular control problem to the primal Problem (3). Then, we obtain the explicit solution to the dual problem. We recover the value function by the duality relationship, by which leads to the optimal polices.

3.1. Singular Control Problem

Let $H_t \equiv e^{-rt}\zeta_t$, where $\zeta_t = e^{-\frac{1}{2}\theta^2 t - \theta B_t}$ with $\theta \equiv (\mu - r)/\sigma$. The wealth process in (2) is rewritten as the static budget constraint as follows.

$$\mathbb{E}\left[\int_0^\infty H_t D_t c_t dt + \int_0^\infty H_t D_t S_t d\mathcal{G}_t^+ - \int_0^\infty H_t D_t (1-\rho) S_t d\mathcal{G}_t^-\right] \\ \leq w_0 + \mathbb{E}\left[\int_0^\infty H_t D_t \phi k_t S_t dt\right], \tag{5}$$

where $\{D_t\}_{t=0}^\infty$ is a positive, non-increasing and right continuous with limits process starting at 1. The left-hand side of the inequality (5) is the sum of the present value of the consumption stream (the first term) and the net value of staked tokens (the sum of the second and third terms). The right-hand side is the sum of the initial wealth and the present value of the reward streams generated by staked tokens.

From the static budget constraint (5), let us define a Lagrangian $\mathfrak{L}$ as follows: for the

Lagrangian multiplier $y > 0$

$$\begin{aligned}\mathfrak{L} :=& \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\omega u(c_t) + (1-\omega)u(k_t S_t)\right) dt\right] + y\left(w + \mathbb{E}\left[\int_0^\infty H_t D_t \phi k_t S_t dt\right]\right.\\ &\left. - \mathbb{E}\left[\int_0^\infty H_t D_t c_t dt + \int_0^\infty H_t D_t S_t d\mathcal{G}_t^+ - \int_0^\infty H_t D_t (1-\rho) S_t d\mathcal{G}_t^-\right]\right)\\ \leq& \mathcal{J}(y, k, s; D, k) + y w_0,\end{aligned} \tag{6}$$

where $\mathcal{J}(y, k, s; D, k)$ is given by

$$\begin{aligned}\mathcal{J}(y, k, s; D, k) =& \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\omega \tilde{u}\left(\frac{Y_t}{\omega}\right) + (1-\omega)u(k_t S_t) + \phi Y_t S_t k_t\right) dt\right.\\ &\left. - \int_0^\infty e^{-\beta t} Y_t S_t d\mathcal{G}_t^+ - (1-\rho)\int_0^\infty e^{-\beta t} Y_t S_t d\mathcal{G}_t^-\right]\end{aligned}$$

with $Y_t := y e^{\beta t} H_t D_t$, and

$$\tilde{u}(z) := \sup_{c>0}\left(u(c) - yc\right) = \frac{\gamma}{1-\gamma} z^{-\frac{1-\gamma}{\gamma}}.$$

Then, we can write down the following *weak-duality*:

$$\begin{aligned}V(w_0, s, k) &\leq \inf_{y>0}\sup_{k_t}\inf_{D_t}\left(\mathcal{J}(y, k, s; \{D_t\}_{t=0}^\infty, \{k_t\}_{t=0}^\infty) + y w_0\right)\\ &\leq \inf_{y>0}\inf_{D_t}\sup_{k_t}\left(\mathcal{J}(y, k, s; \{D_t\}_{t=0}^\infty, \{k_t\}_{t=0}^\infty) + y w_0\right),\end{aligned}$$

where $V(w_0, s, k)$ is the agent's value function defined by (3).

If $\sup_{k_t}\inf_{D_t}\mathcal{J}(y, k, s; \{D_t\}_{t=0}^\infty, \{k_t\}_{t=0}^\infty) = \inf_{D_t}\sup_{k_t}\mathcal{J}(y, k, s; \{D_t\}_{t=0}^\infty, \{k_t\}_{t=0}^\infty)$, we denote this common value by $J(y, k, s)$, i.e.,

$$J(y, k, s) = \sup_{k_t}\inf_{D_t}\mathcal{J}(y, k, s; \{D_t\}_{t=0}^\infty, \{k_t\}_{t=0}^\infty) = \inf_{D_t}\sup_{k_t}\mathcal{J}(y, k, s; \{D_t\}_{t=0}^\infty, \{k_t\}_{t=0}^\infty). \tag{7}$$

By dynamic programming principle, we can derive a Variational Inequality that J satisfies (see (A.1) in the Appendix). Here we consider the following transform:

$$J(y,k,s) = \omega(ks)^{1-\gamma}\mathcal{Q}(z) \;\text{ with }\; z = \frac{y(ks)^{\gamma}}{\omega}. \tag{8}$$

Let us define the following coefficients:

$$\begin{aligned}
\tilde{\omega} &:= (1-\omega)/\omega \\
r_z &:= r - (\mu - \delta) + \sigma\theta = \delta, \\
\beta_z &:= \beta - (1-\gamma)(\mu-\delta) + \frac{\gamma(1-\gamma)\sigma^2}{2}, \\
\sigma_z &:= \gamma\sigma - \theta.
\end{aligned}$$

Then, we can rewrite (A.1) as the following variational inequality for $\mathcal{Q}(z)$: for $z > 0$

$$\begin{cases}
\mathcal{L}_z\mathcal{Q} + \tilde{u}\,(z) + \tilde{\omega}u(1) + \phi z = 0 & \text{if } (1-\rho)z < (1-\gamma)\mathcal{Q} + \gamma z\mathcal{Q}' < z \text{ and } \partial_z\mathcal{Q} < 0, \\
\mathcal{L}_z\mathcal{Q} + \tilde{u}\,(z) + \tilde{\omega}u(1) + \phi z \le 0 & \text{if } (1-\gamma)\mathcal{Q} + \gamma z\mathcal{Q}' = z \text{ or } (1-\gamma)\mathcal{Q} + \gamma z\mathcal{Q}' = (1-\rho)z, \\
\mathcal{L}_z\mathcal{Q} + \tilde{u}\,(z) + \tilde{\omega}u(1) + \phi z \ge 0 & \text{if } \partial_z\mathcal{Q} = 0,
\end{cases} \tag{9}$$

where the operator $\mathcal{L}_z$ is given by

$$\mathcal{L}_z := \frac{\sigma_z^2}{2}z^2\frac{d^2}{dz^2} + (\beta_z - r_z)z\frac{d}{dz} - \beta_z.$$

3.2. Solution to the Free Boundary Problem

In this section we derive the explicit solution to the free boundary problem (9). There are two cases: (i) case for a large ρ (ii) case for a small ρ. Depending on the size of ρ, we have different boundary conditions that specify how the agent chooses the liquid and staked token holdings.

Assumption 1.

$$\beta_z > 0 \ \ and \ \ K := r_z + \frac{\beta_z - r_z}{\gamma} + \frac{(\gamma-1)}{2\gamma^2}\sigma_z^2 > 0.$$

Assumption 1 makes the solution well-defined. Let n_1, n_2 are positive and negative roots of the following quadratic equation:

$$\frac{\theta^2}{2}n^2 + (\beta_z - r_z - \frac{\sigma^2}{2})n - \beta_z = 0. \tag{10}$$

Then, we obtain the explicit form of $\mathcal{Q}(z)$ as in the following proposition.

Proposition 1. *There exists $\rho^* > 0$ such that the following hold:*

(a) If $\rho > \rho^*$,

$$\mathcal{Q}(z) = \hat{C}_1 z^{n_1} + \hat{C}_2 z^{n_2} + \frac{1}{K}\frac{\gamma}{1-\gamma}z^{-\frac{1-\gamma}{\gamma}} + \frac{\tilde{\omega}}{\beta}\frac{1}{1-\gamma} + \frac{\phi}{r}z, \qquad z \in (\hat{z}_H, \hat{z}_L). \tag{11}$$

(b) If $0 < \rho < \rho^*$,

$$\mathcal{Q}(z) = C_1 z^{n_1} + C_2 z^{n_2} + \frac{1}{K}\frac{\gamma}{1-\gamma}z^{-\frac{1-\gamma}{\gamma}} + \frac{\tilde{\omega}}{\beta}\frac{1}{1-\gamma} + \frac{\phi}{r}z, \qquad z \in (z_H, z_L). \tag{12}$$

Here, the coefficients C_1, C_2, $\hat{C}_1$, $\hat{C}_2$ and the free boundaries z_H, z_L $\hat{z}_H$, $\hat{z}_L$ are given in the proof.

As mentioned in the beginning of this subsection, the agent has different token holding policies depending on the size of ρ. Proposition 1 (a) and (b) present the solution forms for each case. In order to precisely describe the optimal policy for each case, we need to define the dual process $\mathcal{Z}_t$ and its behavior for each case as summarized in Corollary 1:

$$\mathcal{Z}_t := \frac{1}{\omega}Y_t(k_t S_t)^\gamma = \frac{y}{\omega}e^{(\beta-r)t}H_t D_t (k_t S_t)^\gamma. \tag{13}$$

Note that $\mathcal{Z}_t$ in (13) has the negative relationship with the wealth process. This is the reason why the lower boundary is denoted by z_H and the upper boundary is denoted by z_L.

Corollary 1. *The optimal polices* (D_t^*, k_t^*) *for Problem* (7) *are given as follows.*

(a) Suppose $\rho > \rho^*$. *Then,* $d\mathcal{G}_t^- = 0$ *for all* $t \geq 0$. *Here,* D_t *and* $\mathcal{G}_t^+$ *are adjusted so* $z_H \leq \mathcal{Z}_t \leq z_L$ *for all* $t \geq 0$. $dk_t^* = -\delta k_t^* dt$ *during the times when* $z_H < \mathcal{Z}_t < z_L$. $\mathcal{Z}_t$ *never hits* z_L *and* k_t^* *increases whenever* $\mathcal{Z}_t$ *hits* z_H.

(b) Suppose $0 < \rho < \rho^*$. *Then,* $\mathcal{G}_t^+$ *and* $\mathcal{G}_t^-$ *are adjusted so that* $z_H \leq \mathcal{Z}_t \leq z_L$ *for all* $t \geq 0$. $D_t^* = 1$ *for all* $t > 0$. $dk_t^* = -\delta k_t^* dt$ *during the times when* $z_H < \mathcal{Z}_t < z_L$. k_t^* *increases (decreases) whenever* $\mathcal{Z}_t$ *hits* z_H (z_L).

When ρ is large, i.e., when the cost from unlocking the staked tokens are substantially high, it is never optimal to unlock the staked tokens ($d\mathcal{G}_t^- = 0$ for all $t \geq 0$). What if the token price drops sufficiently so that wealth becomes closer to zero? In this case, the staked position is completely hedged by shorting the liquid tokens ($x_t < 0$ so that $\pi = x_t^* + m_t^* = 0$) in order to make sure $W_t \geq 0$. This behavior is mathematically described by D_t^* (see the detail in Section 3.3). On the other hand, the ratio of wealth to the staked token holdings, $W_t/k_t^* S_t$, increases sufficiently large, the agent stakes more tokens: the additional amount of newly staked tokens is $S_t d\mathcal{G}_t^+$ whenever $\mathcal{Z}_t$ hits z_H.

Consider the case where ρ is small. In this case, the agent unlocks staked tokens whenever $\mathcal{Z}_t$ hits z_L (or the wealth process hits the zero boundary) and increases staking whenever $\mathcal{Z}_t$ hits z_H (or $W_t/k_t^* S_t$ becomes sufficiently high). In this case, the total token holdings π_t is positive when $W_t = 0$ unlike the case with high ρ and the agent unlocks the staked tokens in order to make $W_t \geq 0$.

3.3. Optimal Strategy

This section provides the optimal strategy to the primal problem (3). To do so, we first establish the following duality relationship between the value function V and the dual value function J.

Theorem 1 (Duality)**.**

$$V(w_0, s, k) = \inf_{y>0} \left(J(y, s, k) + yw_0 \right). \tag{14}$$

From the duality relationship in (14) we have the following condition for optimality: for given $w_0 \geq 0$, $k > 0$, and $s > 0$, there exists a unique $y^* > 0$ such that $y^*(ks)^\gamma/\omega \in (0, z_L)$ and

$$w_0 = -ks\mathcal{Q}'\left(\frac{y^*(ks)^\gamma}{\omega}\right). \tag{15}$$

Proposition 2. *The optimal strategy* (c_t^*, x_t^*, k_t^*) *is given by*

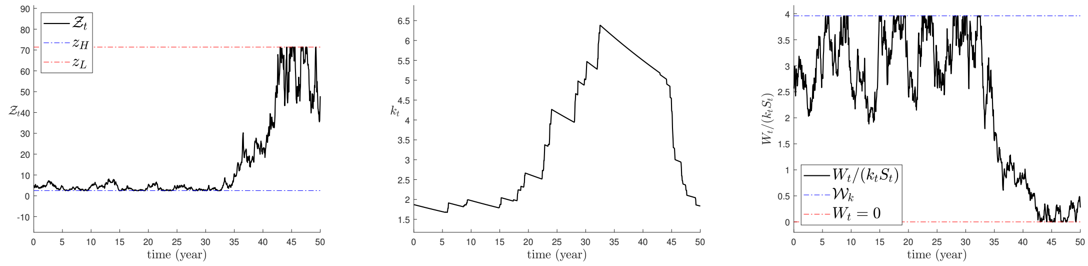
$$\begin{aligned} c_t^* =& m_t(\mathcal{Z}_t^*)^{-\frac{1}{\gamma}}, & (16)\\ x_t^* S_t =& m_t\left(\frac{\theta - \gamma\sigma}{\sigma}\mathcal{Z}_t^*\mathcal{Q}''(\mathcal{Z}_t^*) - \mathcal{Q}'(\mathcal{Z}_t^*)\right) - m_t, & (17) \end{aligned}$$

where m_t *is the total value of the staked token* $m_t := k_t^* S_t$*,* k_t^* *is in Corollary* (1)*, and* $\mathcal{Z}_t$ *is defined by* (13)*. Then, the wealth process* $W_t^{c^*, x_t^*, k_t^*}$ *corresponding to the optimal strategy* (c^*, x^*, k_t^*) *is given by*

$$W_t^{c^*, x^*, k^*} = m_t\left(-\mathcal{Q}'(\mathcal{Z}_t^*)\right). \tag{18}$$

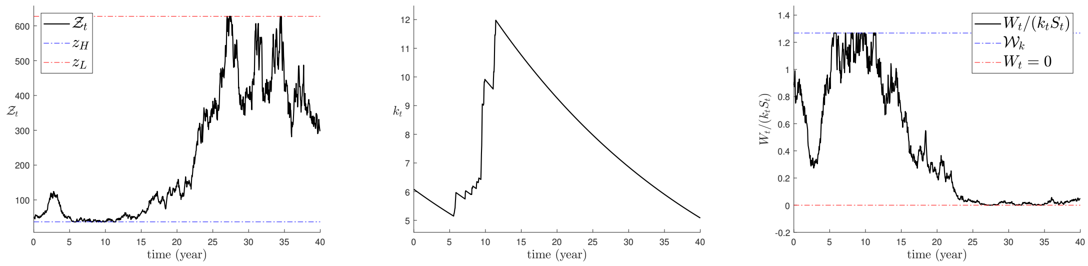
From Theorem 1 and Proposition 2 we can derive the share of the optimal staking boundary. More precisely, for the current staking level k, the current price s, and the total value of staked tokens $m = ks$, the optimal wealth boundary $W_H(m)$ to increase staking and the ratio of maximum wealth to staking $\mathcal{W}_m$ are defined respectively, as follows:

$$W_H(m) := -m\mathcal{Q}'(z_H) \qquad \text{or} \qquad \mathcal{W}_m := \frac{W_H(m)}{m} = -\mathcal{Q}'(z_H). \tag{19}$$



(a) $\mathcal{Z}_t$ (b) k_t^* (c) $\frac{W_t}{m_t}$

Figure 1: Simulation path $\mathcal{Z}_t$, m_t, and $\frac{W_t}{m_t}$ with $\rho = 0.01$. The parameter set is given by $\gamma = 2, \beta = 0.05, \mu = 0.07, \sigma = 0.2, r = 0.03, \omega = 0.3, \phi = 0.02$.



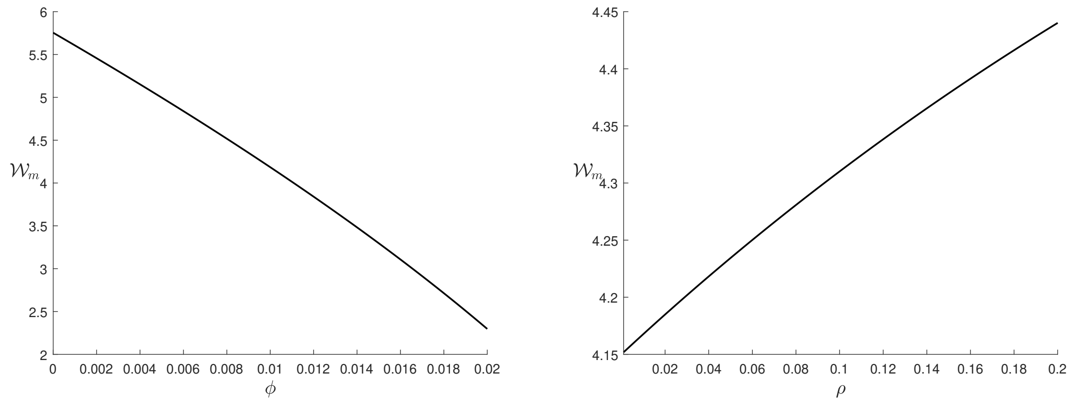
(a) $\mathcal{Z}_t$ (b) k_t^* (c) $\frac{W_t}{m_t}$

Figure 2: Simulation path $\mathcal{Z}_t$, m_t, and $\frac{W_t}{m_t}$ with $\rho = 0.3$. The parameter set is given by $\gamma = 3/2, \beta = 0.05, \mu = 0.07, \sigma = 0.2, r = 0.03, \omega = 0.3, \phi = 0.02$.

Figure 1 and 2 illustrate the simulation path of 50 years for $\mathcal{Z}_t$, k_t^*, and $\frac{W_t}{m_t}$. The boundary z_L (or z_H) corresponds to the wealth level of the minimum wealth to staking ratio $W_t = 0$ (or the maximum wealth to staking ratio $\mathcal{W}_m$). For the case with a small ρ (Figure 1), whenever the dual variable hits z_H (or z_L), the total value of staking m_t decreases (or increases) and the cumulative amount of buying staking tokens $\mathcal{G}_t^+$ (or unlocking staking tokens $\mathcal{G}_t^-$) increases. When ρ is large, however, even when the dual variable $\mathcal{Z}_t$ hits the boundary z_L, the staking amount m_t is unchanged as shown in Figure 2. Instead, the shadow price process D_t decreases whenever $\mathcal{Z}_t$ hits z_L.

4. Implication

4.1. Inaction Interval



(a) $\mathcal{W}_H$ as a function of ϕ (b) $\mathcal{W}_H$ as a function of ρ

Figure 3: The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \phi = 0.01, \rho = 0.02, \omega = 0.5$

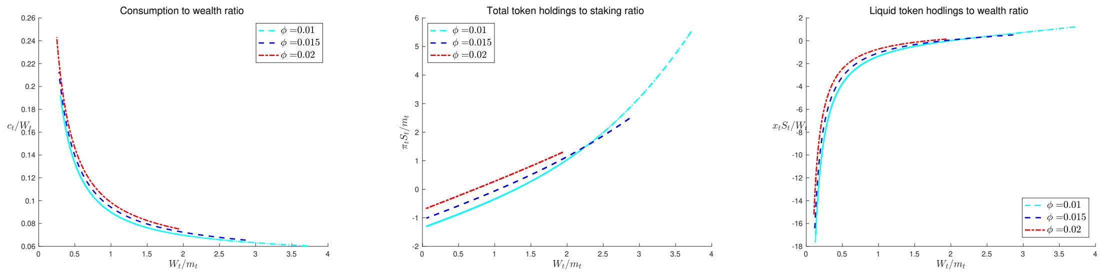
How often does the agent change the staking amount? Proposition3 provides the comparative static result of the size of the inaction region with respect to ϕ and ρ. This result implies that the the staking position is more often changed when ϕ is large and ρ is small. Figure 3 presents $\mathcal{W}_H$ as a function of ϕ (left panel) and $\mathcal{W}_H$ as a function of ρ (right panel).

Proposition 3. *The following hold:*

(a) The inaction region decreases with ϕ: $\frac{\partial \mathcal{W}_H}{\partial \phi} < 0$

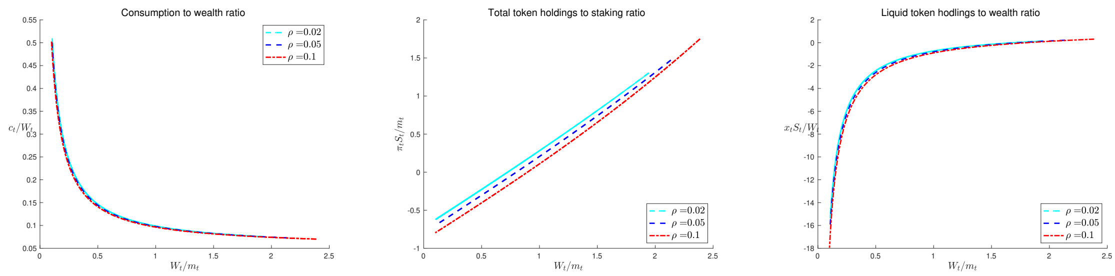
(b) The inaction region increases with ρ: $\frac{\partial \mathcal{W}_H}{\partial \rho} > 0$

Figure 4 and 5 illustrate (a) the optimal consumption to wealth ratio, $\frac{c_t^*}{W_t}$, (b) the ratio of total token holdings to staked token holdings, $\frac{\pi_t S_t}{m_t} = \frac{x_t + k_t}{k_t}$, and (c) the liquid token holdings to wealth ratio $\frac{x_t S_t}{W_t}$ with respect to the wealth to staking ratio, $\frac{W_t}{m_t}$ within an inaction interval. Figures 4(a) and 5(a) show that the ratio of consumption to wealth decreases with wealth



(a) $\frac{c_t}{W_t}$ (b) $\frac{\pi_t}{k_t}$ (c) $\frac{x_t S_t}{W_t}$

Figure 4: Three variables' movement in an inaction interval with different ϕ's. The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \rho = 0.02, \delta = 0.03, \omega = 0.5$



(a) $\frac{c_t}{W_t}$ (b) $\frac{\pi_t}{k_t}$ (c) $\frac{x_t S_t}{W_t}$

Figure 5: Three variables' movement in an inaction interval with different ρ's. The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \phi = 0.02, \delta = 0.03, \omega = 0.5$

while the absolute amount of consumption increases with wealth. This result is immediate from (16) and (18). In contrast, as wealth increases, the total token holdings increase (as Figures 4(b) and 5(b)), which results from the increase in the liquid token holdings (Figures 4(c) and 5(c)).

In summary, during the times when the agent does not stake more tokens nor unlocks the staked tokens, if wealth increases due to the token price increases, the agent spends more funds to buy tokens rather than increases consumption (so that the ratio of consumption to wealth decreases).

4.2. Effect of Staking Reward

Here we investigate the effect of staking reward ϕ. Recall that staking more tokens or unlocking staked tokens happens only when the wealth-to-staking ratio, W_t/m_t hits the upper or lower boundary.

Figure 6 illustrates the effects of staking reward on the consumption to wealth ratio for a given wealth to staking ratio, i.e., fixing $W_t/m_t = 0.1$ (left panel) and $W_t/m_t = 1.5$ (right panel), respectively. That is, the proportion of the staked tokens out of the total token holdings in the left panel is higher than that in the right panel.

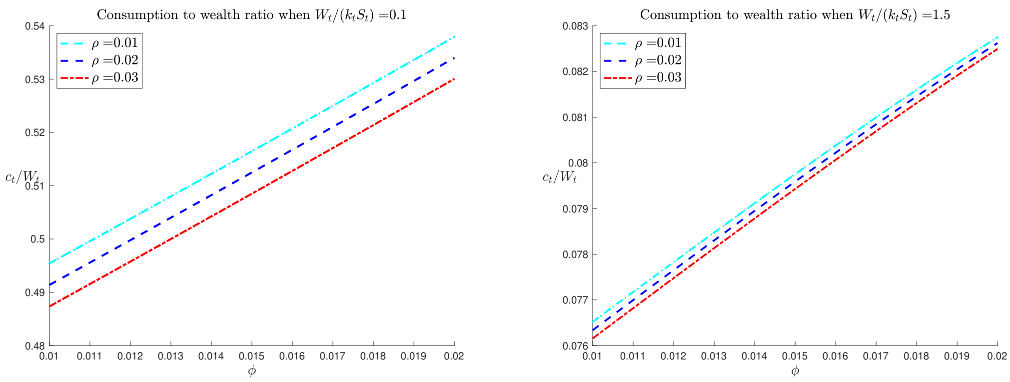
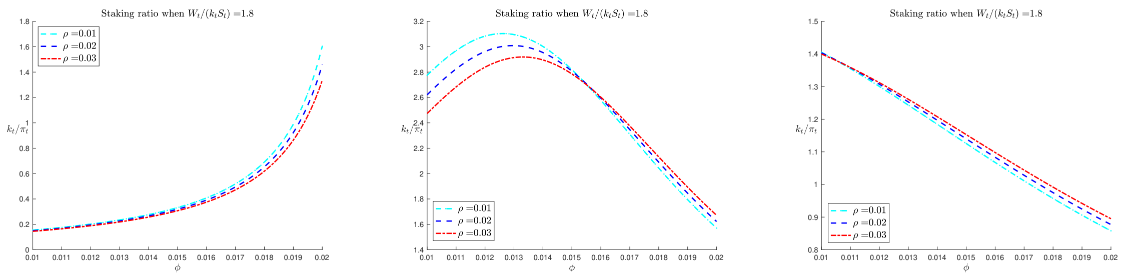


Figure 6: Effect of staking reward (ϕ) on consumption to wealth ratio. The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \delta = 0.03, \omega = 0.5$

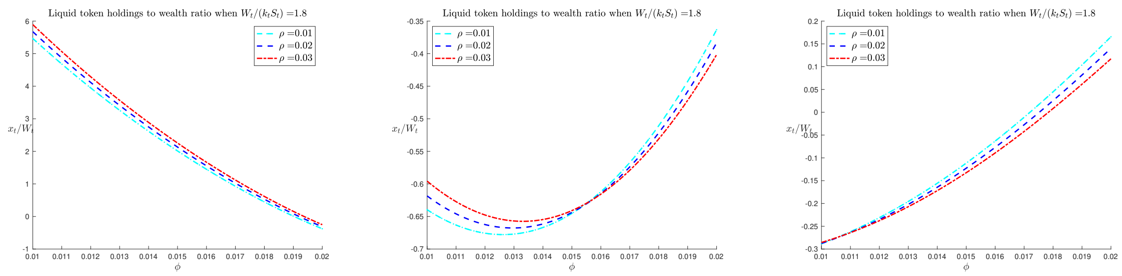
The consumption to wealth ratio increases with the staking reward. In addition, when a proportion of staked tokens is higher, the change in consumption to wealth ratio is more sensitive to the change in the block reward Note the that the scale of y-axis in the right panel is much larger than that in the right panel. This implies that the slope of the graphs in the right panel is much flatter than the graphs in the left panel. In other words, when the agent has a higher staking ratio, he/she can receives a higher reward as ϕ increases and thus increases consumption further as ϕ increases.

The effect of the cost ρ is negative on consumption: as the cost of unlocking the staked tokens increases, consumption decreases (compare three graphs with $\rho = 0.01, 0.02$, and 0.03 in Figure 6. The reason is that as the cost increases, the net return from investment goes down so that consumption decreases.



(a) ω=0.9 (b) ω=0.65 (c) ω=0.5

Figure 7: Effect of ϕ on the staking ratio, $k_t/(x_t + k_t)$, when $W_t/m_t = 1.8$ (fixed). The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \delta = 0.03$



(a) ω=0.9 (b) ω=0.65 (c) ω=0.5

Figure 8: Effect of ϕ on liquid token to wealth ratio, x_tS_t/W_t, when $W_t/m_t = 1.8$ (fixed). The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \delta = 0.03$

What about the impact of the staking reward on the staking ratio, $\frac{k_t}{k_t+x_t}$? One might think that the staking ratio increases as the reward increases. This is only true when the agent does not have much utility from staking tokens. In this case, as seen in Figure 7(a) with $\omega = 0.9$), the agent increases the staking ratio in order to obtain the reward from staking as ϕ increases. However, if ω is small so that the agent obtain large utility from staking tokens, the result is overturned: the staking ratio decreases with the staking reward as seen in Figure 7(c).

To understand the above result, we first note the staking reward can be considered as the additional income stream and thus the increase in the reward effectively increase the total wealth, which is the sum of current wealth and the present value of income. Therefore, both x_t and k_t increase with ϕ. In other words, the agent optimally increases total token investment as the staking reward increases due to the wealth effect. Therefore, the reason why the staking ratio increases (decreases) with ϕ when ω is small (large) is that the increase in the liquid token holdings is greater (smaller) than the increase in the stake token holdings as ϕ increases.

With the above mechanism in mind, as ω decreases (the agent' value from staking increases), the ratio of consumption to the agent's staking, $c_t/k_t^* S_t$ decreases, which leads to the increase in the marginal utility from consumption. Therefore, when ω is small, as the agent increases the total token investment π_t, liquid token holdings increase more than the staked holdings do in order to have a higher liquid wealth position to smooth out consumption in response to a bad shock. Otherwise, the agent should liquidate the staked tokens against a bad shock, which incur additional cost from unlocking tokens.

4.3. Effect of the Cost from Unlocking the Staked Token

Figure 9 illustrates the effect of ρ on the staking ratio when the wealth to staking ratio is fixed. Regardless of the level of the wealth to staking ratio and the staking reward, the staking ratio always increases with the locking cost.

5. Conclusion

In this paper, we considered the optimal staking decision problem of the agent who participates in a DAO using a governance token. The agent receives the benefits of staking token

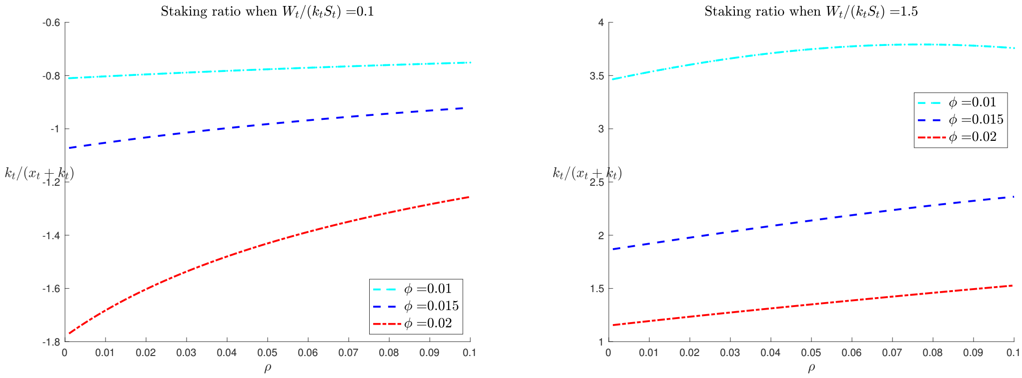


Figure 9: Effects of the lockup cost (ρ) on staking ratio ($k_t/(x_t + x_t)$). The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \delta = 0.03, \omega = 0.5$.

holdings from the staking reward and utility gain. Instead, he/she faces a unlocking cost. Then, the optimal staking policy is that the agent never changes the staking position until wealth reaches zero or a sufficient high level. We investigated various implications of the staking reward and the cost for the optimal policies.

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A. Appendix

A.1. Variabtional Inequality for J

By applying the dynamic programming principle, we can expect that $J(y,k,s)$ satisfy the following variational inequality:

$$\begin{cases} \mathcal{L}_{y,s,k}J + \omega\tilde{u}\left(\dfrac{y}{\omega}\right) + (1-\omega)u(k) + \phi ysk = 0 & \text{if } (1-\rho)ys < \partial_k J < ys \text{ and } \partial_y J < 0, \\ \mathcal{L}_{y,s,k}J + \omega\tilde{u}\left(\dfrac{y}{\omega}\right) + (1-\omega)u(k) + \phi ysk \leq 0 & \text{if } \partial_k J = ys \text{ or } \partial_k J = (1-\rho)ys, \\ \mathcal{L}_{y,s,k}J + \omega\tilde{u}\left(\dfrac{y}{\omega}\right) + (1-\omega)u(k) + \phi ysk \geq 0 & \text{if } \partial_y J = 0, \end{cases} \tag{A.1}$$

where $\mathcal{L}_{y,s,k}$ is given by

$$\mathcal{L}_{y,s,k} := \frac{\theta^2}{2}y^2\partial_{yy} + \frac{\sigma^2}{2}s^2\partial_{ss} - \theta\sigma\partial_{sy} + (\beta - r)y\partial_y + \mu s\partial_y - \delta k\partial_k - \beta.$$

Note that under transform (8), the conditions $\partial_k J = 0$ and $\partial_y J = 0$ are equivalent to $(1-\gamma)\mathcal{Q} + \gamma z\mathcal{Q}' = 0$ and $\mathcal{Q}' = 0$, respectively.

B. Proof of Proposition 1

For (a): If $\rho > \rho^*$, we consider the following free boundary problem:

$$\mathcal{L}_z\mathcal{Q} + \tilde{u}(z) + \tilde{\omega}u(1) + \phi z = 0 \text{ for } \hat{z}_H < z < \hat{z}_L, \tag{B.1}$$

$$(1-\gamma)\mathcal{Q}(\hat{z}_H) + \gamma\hat{z}_H\mathcal{Q}'(\hat{z}_H) = \hat{z}_H, \quad \mathcal{Q}'(\hat{z}_H) + \gamma\hat{z}_H\mathcal{Q}''(\hat{z}_H) = 1, \tag{B.2}$$

$$\mathcal{Q}'(\hat{z}_L) = 0 \quad \mathcal{Q}''(\hat{z}_L) = 0. \tag{B.3}$$

For $\hat{z}_H < z < \hat{z}_L$, we write a general solution of $\mathcal{Q}$ is given as (11).

By using the boundary condition (B.2) and (B.3), we have

$$\hat{C}_1 = -\frac{\hat{z}_H^{-n_1}}{(n_2-n_1)}\left[\phi\frac{(n_2-1)}{r}\hat{z}_H + \frac{\tilde{\omega}n_2}{\beta}\right] \text{ and } \hat{C}_2 = -\frac{\hat{z}_H^{-n_2}}{(n_1-n_2)}\left[\phi\frac{(n_1-1)}{r}\hat{z}_H + \frac{\tilde{\omega}n_1}{\beta}\right]. \tag{B.4}$$

Moreover,

$$\hat{z}_H = q(\xi) \text{ and } \hat{z}_L = \xi q(\xi),$$

where

$$q(\xi) := \left[\frac{n_1 n_2 \tilde{\omega}}{\beta}(\xi^{n_2} - \xi^{n_1})\right] \times \left[\left(\frac{\phi}{r} - 1\right)(\xi^{n_1} n_1(n_2-1) - \xi^{n_2} n_2(n_1-1)) - (n_2-n_1)\frac{\phi}{r}\xi\right]^{-1}.$$

and $\xi \in (1,\infty)$ satisfies

$$\frac{n_1 n_2 \tilde{\omega}}{\beta}(\xi^{n_2} - \xi^{n_1}) = \left(\frac{\phi}{r} - 1\right)(\xi^{n_1} n_1(n_2-1) - \xi^{n_2} n_2(n_1-1))\, q(\xi) - (n_2-n_1)\frac{\phi}{r}\xi q(\xi).$$

For (b): If $\rho < \rho^*$, we consider the following free boundary problem:

$$\mathcal{L}_z\mathcal{Q} + \tilde{u}(z) + \tilde{\omega}u(1) + \phi z = 0 \text{ for } z_H < z < z_L, \tag{B.5}$$

$$(1-\gamma)\mathcal{Q}(z_H) + \gamma z_H \mathcal{Q}'(z_H) = z_H, \quad \mathcal{Q}'(z_H) + \gamma z_H \mathcal{Q}''(z_H) = 1, \tag{B.6}$$

$$(1-\gamma)\mathcal{Q}(z_L) + \gamma z_L \mathcal{Q}'(z_L) = (1-\rho) z_L, \quad \mathcal{Q}'(z_L) = 0. \tag{B.7}$$

For $z_H < z < z_L$, we write a general solution of $\mathcal{Q}$ is given as (12).

By using the boundary condition (B.6) and (B.7), we have

$$C_1 = -\frac{z_H^{-n_1}}{(n_2-n_1)}\left[\phi\frac{(n_2-1)}{r}z_H + \frac{\tilde{\omega}n_2}{\beta}\right] \text{ and } C_2 = -\frac{z_H^{-n_2}}{(n_1-n_2)}\left[\phi\frac{(n_1-1)}{r}z_H + \frac{\tilde{\omega}n_1}{\beta}\right]. \tag{B.8}$$

and $Z_H = q(\xi)$ and $z_L = \xi q(\xi)$, where

$$q(\xi) := \left[\frac{\tilde{\omega}}{\beta}\left(\frac{n_2}{n_2 - n_1}\xi^{n_1} + \frac{n_1}{n_1 - n_2}\xi^{n_2} - 1\right)\right]$$
$$\times \left[-(\phi - r)\frac{(n_2 - 1)}{(n_2 - n_1)r}\xi^{n_1} - (\phi - r)\frac{(n_1 - 1)}{(n_1 - n_2)r}\xi^{n_2} + \frac{(\phi - (1 - \rho))}{r}\xi\right]^{-1}.$$

and $\xi \in (1, \infty)$ satisfies

$$\frac{\tilde{\omega}}{\beta}\left(\frac{n_2}{n_2 - n_1}\xi^{n_1} + \frac{n_1}{n_1 - n_2}\xi^{n_2} - 1\right)$$
$$= -(\phi - r)\frac{(n_2 - 1)}{(n_2 - n_1)r}\xi^{n_1}q(\xi) - (\phi - r)\frac{(n_1 - 1)}{(n_1 - n_2)r}\xi^{n_2}q(\xi) + \frac{(\phi - (1 - \rho))}{r}\xi q(\xi).$$